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Well-posedness for vector equilibrium problems

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Abstract We introduce and study two notions of well-posedness for vector equilibrium problems in topological vector spaces; they arise from the well-posedness concepts previously introduced by the same authors in the scalar case, and provide an extension of similar definitions for vector optimization problems. The first notion is linked to the behaviour of suitable maximizing sequences, while the second one is defined in terms of Hausdorff convergence of the map of approximate solutions. In this paper we compare them, and, in a concave setting, we give sufficient conditions on the data in order to guarantee well-posedness. Our results extend similar results established for vector optimization problems known in the literature.

Keywords Well-posedness · Vector equilibrium problems · Approximate solutions

Mathematics Subject Classification (2000) 49K40 · 90C31

1 Introduction

Let X and Y be topological vector spaces with countable local bases, and K be a closed convex cone in Y with nonempty interior. Given $f : D \times D \to Y$, with $D \subseteq X$ and

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$f(x, x) = 0$, for all $x \in D$, the vector equilibrium problem is: find $\overline{x} \in D$ such that

$$f(\overline{x}, y) \notin -K_0, \quad \forall y \in D, \tag{VEP}$$

where K_0 denotes the set $K_0 = K \setminus \{0\}$. We call S the solution set and we will suppose in the sequel that S is nonempty.

It is well known that vector equilibrium problems are natural extensions of several problems of practical interest like vector optimization and vector variational inequality problems. Our purpose is to try to assign reasonable definitions of well-posedness to (VEP) that recover some previous existing concepts (see, for instance, Bednarczuk 1994; Miglierina and Molho 2003; Miglierina et al. 2005).

Recall that the scalar equilibrium problem is the particular instance of (VEP) (denoted (EP)) with $Y = \mathbb{R}$ and $K = [0, +\infty)$. It is easy to see that (EP) admits solutions if and only if $\max_{a \in D} g(a) = 0$, where $g : D \to [-\infty, +\infty)$ is the gap function defined by $g(a) = \inf_{b \in D} f(a, b)$. In a previous paper where the scalar equilibrium problem was discussed (see Bianchi et al. 2007), the authors studied Tykhonov well-posedness defined via a minimax formulation of the equilibrium problem.

We start our analysis following a similar approach. To this aim, we introduce the set valued map $\phi : D \to 2^Y$ given by

$$\phi(x) = \min_K (f(x, D)) \tag{1.1}$$

(see also Ansari et al. 2002), where for any $A \subseteq Y$, the (possibly empty) set of minimal elements is defined as follows:

$$\min_K(A) = \{a' \in A : \ (A - a') \cap (-K_0) = \emptyset\}.$$

The map ϕ generalizes the definition of the function g; in particular, the solutions can be characterized in terms of ϕ since $\overline{x} \in S$ if and only if $0 \in \phi(\overline{x})$. Through the map ϕ, we can define maximizing sequences and approximate solutions, that, as it is well-known, are key concepts in the investigation of well-posedness.

The paper is organized as follows. In Sect. 2 two notions of well-posedness are introduced; these notions, in case of optimization problems, reduce to the ones considered in Bednarczuk (1994) and in Miglierina and Molho (2003). Section 3 is devoted to the comparison between these two different definitions; as a matter of fact, we show that one of them is stronger. In Sect. 4 sufficient conditions for well-posedness in a concave setting are proved; in particular, Theorem 1 provides an extension of Theorem 5.1 in Miglierina and Molho (2003).

2 Well-posed problems

In the next proposition some properties of ϕ are pointed out; in particular, assuming that the solution set is nonempty, we obtain that $\text{dom}(\phi) \neq \emptyset$.

Proposition 1 *The map ϕ satisfies the relations:*

(i) $\phi(x) \cap K_0 = \emptyset$, *for all* $x \in D$;
(ii) $\overline{x} \in S \Longleftrightarrow 0 \in \phi(\overline{x})$;
(iii) $\overline{x} \in S \Longleftrightarrow \phi(\overline{x}) \cap K \neq \emptyset$.

Proof (i) Assume that for some $x' \in D$, $\phi(x') \cap K_0 \neq \emptyset$. Then there exists $y' \in K_0$ such that $y' \in \min_K f(x', D)$, that is equivalent to say that $(f(x', D) - y') \cap (-K_0) = \emptyset$. Since $0 \in f(x', D)$, we get that $\{-y'\} \cap (-K_0) = \emptyset$, a contradiction.
(ii) indeed, taking into account that $0 \in f(x, D)$ for every $x \in D$,

$$\overline{x} \in S \Longleftrightarrow f(\overline{x}, y) \notin (-K_0), \forall y \in D \Longleftrightarrow f(\overline{x}, D) \cap (-K) = \{0\};$$

this is equivalent to say that $0 \in \min_K f(\overline{x}, D) = \phi(\overline{x})$.
(iii) trivial, by (i) and (ii). □

In the sequel, we shall denote by $\mathcal{V}_X(x_0)$ a neighborhood base of x_0 in the topological vector space X. The same notation will be used for other spaces.

Let us recall the following notion of upper Hausdorff convergence of a sequence of points to a set (see, for instance, Miglierina and Molho 2003).

Definition 1 The sequence $\{x_n\} \subset X$ is said to be *upper Hausdorff convergent* to the set $A \subset X$ ($x_n \xrightarrow{H} A$) if for every neighborhood $V_X \in \mathcal{V}_X(0)$ there exists n_0 such that $x_n \in A + V_X$, for every $n \geq n_0$.

To introduce the first notion of well-posedness for (VEP), we need the following:

Definition 2 A sequence $\{x_n\} \subset D$ is said to be a *maximizing sequence for* ϕ if for every $V_Y \in \mathcal{V}_Y(0)$, there exists $n_0 \in \mathbb{N}$ such that

$$\phi(x_n) \cap V_Y \neq \emptyset, \quad \forall n \geq n_o.$$

Definition 2 is related to Definition 4.1 in Miglierina and Molho (2003) in case of vector optimization, where A_n is a singleton. Indeed, the following proposition holds:

Proposition 2 *If* $f(x, y) = F(y) - F(x)$, *then* $\{x_n\}$ *is maximizing if and only if*

$$F(x_n) \xrightarrow{H} \min_K F(D),$$

i.e., $\{x_n\}$ *is a minimizing sequence for the vector optimization problem, according to Miglierina and Molho (2003).*

Proof Since Y is a topological vector space, we can always choose a base of radial, balanced neighborhoods $\mathcal{V}_Y(0)$ of 0 (see Aliprantis and Border 1999). In particular, for every $V_Y \in \mathcal{V}_Y(0)$, if $z \in V_Y$, then $-z \in V_Y$.

$\Longrightarrow$ Fix $V_Y \in \mathcal{V}_Y(0)$; from Definition 2, there exists $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$,

$$\phi(x_n) \cap V_Y = (\min_K F(D) - F(x_n)) \cap V_Y \neq \emptyset,$$

that is, there exists $y \in \min_K F(D)$ such that $F(x_n) \in y + V_Y$, if $n \geq n_0$.

$\Longleftarrow$ Take an arbitrary $V_Y \in \mathcal{V}_Y(0)$; for $n \geq n_0$, $F(x_n) \in \min_K F(D) + V_Y$, that is there exists $y \in \min_K F(D)$ such that $y - F(x_n) \in V_Y$, thereby proving that $\phi(x_n) \cap V_Y \neq \emptyset$. □

The next definition reproduces, in the vector setting, the classical notion of Tykhonov well-posedness given in metric spaces (see, for instance, Miglierina et al. 2005, Definition 3.7), and it generalizes the definition of $\mathrm{T}_{\mathrm{opt}}$-well-posedness given in Bianchi et al. (2007).

Definition 3 We say that the vector equilibrium problem (VEP) is *M-well-posed* if

(i) there exists at least one solution, i.e., $S \neq \emptyset$;
(ii) for every maximizing sequence, and for every $V_X \in \mathcal{V}_X(0)$, there exists n_0 such that $x_n \in S + V_X$, for every $n \geq n_0$.

Another notion of well-posedness can be given in terms of regularity of a suitable approximate solution map. This approach, proposed in Bednarczuk (1994) for vector optimization problems, has been already exploited by several authors (see, for instance, Miglierina and Molho 2003; Miglierina et al. 2005, and the references therein).

In the previous paper Bianchi et al. (2007) the authors introduced and compared different notions of approximate solutions in the scalar case. One of these is given via the notion of ε-argmin(EP) points, i.e. the set of points $x \in D$ such that $f(x, y) \geq -\varepsilon$, for every $y \in D$.

In the sequel, we extend the definition of ε-argmin(EP) to the vector valued case.

Definition 4 Given $\varepsilon \in K$, the set

$$S(\varepsilon) = \{x \in D : \phi(x) \cap (K - \varepsilon) \neq \emptyset\}$$

is called the ε-*approximate solution set* of (VEP).

Notice that $S(0) = S$, by (iii) in Proposition 1.

Remark 1 The definition above is also related to the notion of ε-minimal solutions

$$Q(\varepsilon) = \cup_{y \in \min_K F(D)} \{x \in D : \ F(x) \in y + \varepsilon - K\}.$$

introduced in Bednarczuk (1994) (see also Miglierina and Molho 2003). Indeed, in case of vector optimization problems, where $f(x, y) = F(y) - F(x)$, one trivially shows that $S(\varepsilon) = Q(\varepsilon)$: for every $x \in D$, $\phi(x) = \min_K F(D) - F(x)$, and

$$\begin{aligned} x \in S(\varepsilon) &\Longleftrightarrow \exists y \in \min_K F(D) : \ y - F(x) \in K - \varepsilon \\ &\Longleftrightarrow \exists y \in \min_K F(D) : \ F(x) \in y + \varepsilon - K \\ &\Longleftrightarrow x \in \cup_{y \in \min_K F(D)} \{x' \in D : \ F(x') \in y + \varepsilon - K\} \\ &\Longleftrightarrow x \in Q(\varepsilon). \end{aligned}$$

The next definition assumes some continuity of the map $S(\cdot)$, namely its upper Hausdorff continuity. Let us recall that a map $T : Z \to 2^W$, with Z, W topological spaces, is said to be *upper semicontinuous* at z_0 if for every neighborhood U of $T(z_0)$, there exists a neighborhood V of z_0 such that

$$T(z) \subseteq U, \quad \forall z \in V$$

(see Aubin and Frankowska 1984). In case W is also a vector space, the notion above can be weakened by requiring that the arbitrary neighborhood of $T(z_0)$ is of the form $T(z_0) + V_W$, where $V_W \in \mathcal{V}_W(0)$. In this case we say that the map is *upper Hausdorff continuous*.

Definition 5 We say that the vector equilibrium problem (VEP) is *B-well-posed* if

(i) there exists at least one solution, i.e., $S \neq \emptyset$;
(ii) the map $S(\cdot) : K \to 2^X$ is upper Hausdorff continuous at $\varepsilon = 0$, i.e., for every $V_X \in \mathcal{V}_X(0)$ there exists $V_Y \in \mathcal{V}_Y(0)$ such that $S(\varepsilon) \subset S + V_X$ for every $\varepsilon \in V_Y \cap K$.

We would like to point out that the stronger assumption of upper semicontinuity of $S(\cdot)$ appears to be too restrictive, because, in this case, well-behaving problems would not be B-well-posed (see Example 3.1 in Miglierina and Molho 2003).

3 Relationship between M and B-well-posedness

We shall investigate in this section the relationship between the two concepts of well-posedness for (VEP) introduced in the previous section. It turns out that B-well-posedness always implies M-well-posedness, but the converse is not true in general. However, we find a suitable condition under which the converse also holds.

The next proposition extends a similar result in Miglierina and Molho (2003).

Proposition 3 *Any B-well-posed vector equilibrium problem is M-well-posed.*

Proof By contradiction, suppose that there exists a maximizing sequence $\{x_n^*\}$ and a neighborhood $V_X^* \in \mathcal{V}_X(0)$ such that

$$x_n^* \notin S + V_X^* \text{ for infinitely many } n\text{-s.} \tag{3.1}$$

Let $\varepsilon_n \in \text{int}K$ such that $\varepsilon_n \to 0$. Then, for every $n \in \mathbb{N}$, there exists $V_n \in \mathcal{V}_Y(0)$ such that $V_n \subseteq K - \varepsilon_n$. Since $\{x_n^*\}$ is maximizing, for every $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}$ such that, if $m \geq m_n$,

$$\phi(x_m^*) \cap V_n \neq \emptyset;$$

in particular, $\phi(x_m^*) \cap (K - \varepsilon_n) \neq \emptyset$, i.e., $x_m^* \in S(\varepsilon_n)$, for all $m \geq m_n$. From the B-well-posedness assumption, there exists $n^* \in \mathbb{N}$ such that, for $n \geq n^*$,

$$S(\varepsilon_n) \subseteq S + V_X^*,$$

a contradiction. □

The converse does not hold, even in the particular case of vector optimization, unless some assumptions are added. Indeed, in Miglierina and Molho (2003), an example is provided showing that M-well-posedness does not imply B-well-posedness.

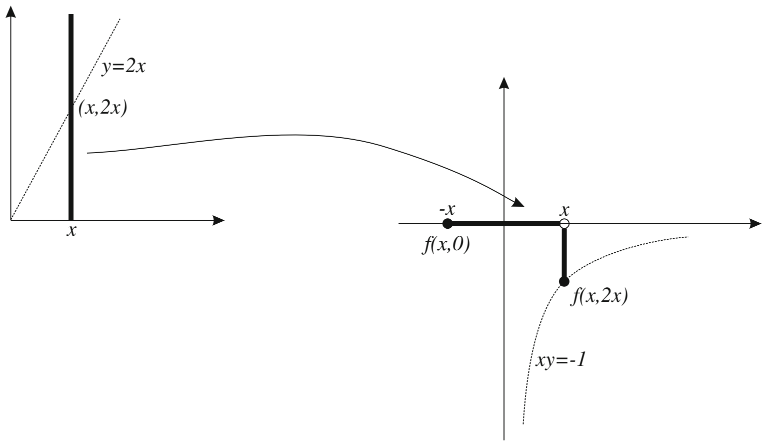
In the next example we show that also for those vector equilibrium problems, that cannot be reduced to an optimization problem, B-well-posedness is stronger than M-well-posedness.

Example 1 Let $K = \mathbb{R}^2_+$ and $f : [0, +\infty) \times [0, +\infty) \to \mathbb{R}^2$ be given by the following rule:

- For $x = 0$ put $f(x, y) = (0, 0), \quad \forall y \geq 0.$
- For $x > 0$ let

$$f(x, y) = \begin{cases} (y - x, 0), & 0 \leq y < 2x \\ (x, -\frac{4x}{y^2}), & y \geq 2x. \end{cases}$$

The picture below explains how the function f acts on a point $(x, y) \in [0, +\infty) \times [0, +\infty)$:



Obviously $f(x, x) = (0, 0)$ for every $x \geq 0$. It is easy to check that the solution set of the associated vector (VEP) is $S = \{0\}$. Let us first show that (VEP) is M-well-posed. To do this, we shall evaluate the set-valued mapping ϕ. We have

$$\phi(x) = \begin{cases} \{(0, 0)\}, & x = 0 \\ \{(-x, 0), \left(x, -\frac{1}{x}\right)\}, & x > 0. \end{cases}$$

Let $\{x_n\}$ be an arbitrary maximizing sequence. Then, by definition, we have that $\{x_n\} \to 0$ as $n \to +\infty$, which shows that our (VEP) is M-well-posed.

Now let us prove that (VEP) is not B-well-posed. Take for this the neighborhood V^*_X of 0 such that $V^*_X \cup D = [0, 1/2)$, and the sequences $\varepsilon_n = \left(0, \frac{1}{n}\right)$, $x_n = n$ for every $n \geq 1$. Then obviously $x_n \in S(\varepsilon_n)$, but $x_n \notin S + V_X^* = V_X^*$ for every $n \geq 1$, showing that (VEP) is not B-well-posed.

In the next proposition we give a sufficient condition that enables us to prove the converse implication.

Proposition 4 *Assume that the vector equilibrium problem is M-well-posed and for every $V_Y \in \mathcal{V}_Y(0)$ there exists $\tilde{V}_Y \in \mathcal{V}_Y(0)$ such that*

$$\phi(D \backslash cl(S)) \cap (K + \tilde{V}_Y) \subseteq V_Y. \tag{3.2}$$

Then, the problem is B-well-posed.

Proof Suppose, by contradiction, that there exist a neighborhood V_X^* of the origin, a sequence $\{\varepsilon_n\} \subset K$, $\varepsilon_n \to 0$ and $x_n \in S(\varepsilon_n)$ such that

$$x_n \notin S + V_X^*, \quad \forall n \in \mathbb{N};$$

besides, this implies that $x_n \notin cl(S)$. If $\{x_n\}$ is a maximizing sequence, the contradiction is trivial by M-well-posedness. Otherwise, there exist $V_Y^* \in \mathcal{V}_Y(0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\phi(x_{n_k}) \cap V_Y^* = \emptyset, \quad \forall k \in \mathbb{N}.$$

Since $x_{n_k} \in S(\varepsilon_{n_k})$, we know that

$$\phi(x_{n_k}) \cap (K - \varepsilon_{n_k}) \neq \emptyset.$$

Let $\tilde{V}_Y^*$ be the neighborhood related to V_Y^* in (3.2). Then, for k large enough, $-\varepsilon_{n_k} \in \tilde{V}_Y^*$. Take $\overline{y}_{n_k} \in \phi(x_{n_k}) \cap (K - \varepsilon_{n_k})$. We have that $\overline{y}_{n_k} \in K + \tilde{V}_Y^*$, which, together with (3.2), leads to the contradiction $\overline{y}_{n_k} \in V_Y^*$. This completes the proof. □

4 Concavity and well-posedness

It is well known that both the assumptions of concavity and monotonicity together with their generalizations are strictly related to many results concerning equilibrium problems. In this section we discuss the role of these conditions in order to single out classes of well-posed vector equilibrium problems. In particular, we focus on concave functions, by providing results that extend a similar one in Miglierina and Molho (2003) in the case of vector optimization.

Let us recall that a function $h : D \subseteq X \to Y$, with D convex is said to be K-concave if

$$\lambda h(x) + (1 - \lambda)h(y) \in h(\lambda x + (1 - \lambda)y) - K,$$

for every $x, y \in D$, and for every $\lambda \in [0, 1]$. A function $h : X \to Y$ is said to be K-upper continuous at $x_0 \in X$ if for each $V_Y \in \mathcal{V}_Y(0)$ there exists $V_X \in \mathcal{V}_X(x_0)$ such that

$$h(V_X) \subset h(x_0) + V_Y - K$$

(see, for instance, Bednarczuk 1994).

We will suppose in the sequel that the topological vector space Y is regular, i.e., every nonempty closed set and every singleton disjoint from it can be separated by open sets.

Theorem 1 *Let X be finite dimensional ($X = \mathbb{R}^n$), $D \subseteq X$ be a closed convex set, and $f : D \times D \to Y$ such that*

- (i) *the solution set S of (VEP) is nonempty and bounded;*
- (ii) *the map $\phi : D \to 2^Y$ is upper Hausdorff continuous with closed values;*
- (iii) *$f(x, z) \neq 0$ whenever $x \in S$ and $z \in D \backslash S$;*
- (iv) *the function $x \mapsto f(x, z)$ is K-upper continuous and K-concave, for every $z \in D$;*
- (v) *for every maximizing sequence $\{x_n\} \subset D$ and every $z \in D$, the sequence $\{f(x_n, z)\}$ is bounded in Y.*

Then the problem (VEP) is M-well-posed.

Proof Suppose by contradiction that there exist a maximizing sequence $\{x_n\}$ and $\varepsilon > 0$ such that

$$x_n \notin S + \varepsilon B, \tag{4.1}$$

for infinitely many n's, where B denotes the unit open ball in X. Since every subsequence of a maximizing sequence is still maximizing, we may assume for simplicity that relation (4.1) holds for every n. Let us distinguish the following two situations:

1. The sequence $\{x_n\}$ is bounded, hence it has a convergent subsequence $\{x_{n_k}\}$ with limit $x^* \in D$. Assume that $x^* \notin S$, that is $0 \notin \phi(x^*)$. The set $\phi(x^*)$ is closed, nonempty, and does not contain 0; in particular, since Y is regular, there exists $V_Y \in \mathcal{V}_Y(0)$ such that

$$\big(\phi(x^*) + V_Y\big) \cap V_Y = \emptyset. \tag{4.2}$$

Since $\{x_n\}$ is maximizing, one can choose a sequence $y_n \in \phi(x_n)$ with $y_n \to 0$. By the upper Hausdorff continuity of ϕ at x^*, we have that, definitely,

$$\phi(x_n) \subset \phi(x^*) + V_Y,$$

but this contradicts (4.2).

2. The sequence $\{x_n\}$ is unbounded. Since S is bounded, we obtain that $S + \varepsilon B$ is bounded, thus the set $\mathrm{cl}(S + \varepsilon B)$ is compact. Consider the compact set $Q = \mathrm{bd}(S+\varepsilon B) = \mathrm{cl}(S+\varepsilon B)\backslash \mathrm{int}(S+\varepsilon B)$. For any $x \in S$ we have $x+\varepsilon B \subseteq S+\varepsilon B$,

therefore $S \subseteq \operatorname{int}(S + \varepsilon B)$, showing that $S \cap Q = \emptyset$. By the hypothesis, $\{x_n\}$ admits a subsequence (denoted for convenience also by $\{x_n\}$) converging in norm to $+\infty$. Let us suppose (without loss of generality) that $x_n \notin \operatorname{cl}(S+\varepsilon B)$ for every n. Fix an arbitrary $\bar{x} \in S$ and for any n, let $x'_n = \lambda_n \bar{x} + (1-\lambda_n)x_n \in Q$ where

$$\lambda_n := \sup\{\lambda \in [0,1] : \lambda\bar{x} + (1-\lambda)x_n \notin S + \varepsilon B\}.$$

It is easy to check that $\lambda_n \to 1$. Indeed, if not, there exists some $\delta < 1$ such that $\lambda_n \leq \delta$ for infinitely many n's. Thus we may write

$$(1-\lambda_n)x_n = x'_n - \lambda_n \bar{x},$$

from which

$$\|x_n\| \leq \frac{1}{1-\lambda_n}\|x'_n\| + \frac{\lambda_n}{1-\lambda_n}\|\bar{x}\| \leq \frac{1}{1-\delta}(\|x'_n\| + \|\bar{x}\|),$$

for infinitely many n's. But this contradicts the fact that $\|x_n\| \to +\infty$.

The set Q being compact, we can extract from the sequence $\{\lambda_n \bar{x} + (1-\lambda_n)x_n\}$ a subsequence $\{\lambda_{n_k}\bar{x} + (1-\lambda_{n_k})x_{n_k}\}$ converging to $x' \in Q$. Then by concavity we obtain for every $k \in \mathbb{N}$:

$$\lambda_{n_k} f(\bar{x}, x') + (1-\lambda_{n_k}) f(x_{n_k}, x') \in f(\lambda_{n_k}\bar{x} + (1-\lambda_{n_k})x_{n_k}, x') - K. \quad (4.3)$$

Observe the sum in the left-hand side still converges to $f(\bar{x}, x')$. We show that

$$f(\bar{x}, x') \in -K. \quad (4.4)$$

Indeed, let us suppose the contrary: $f(\bar{x}, x') \notin -K$. Then there exists $V_Y \in \mathcal{V}_Y(0)$ such that

$$(V_Y - K) \cap (f(\bar{x}, x') + V_Y) = \emptyset. \quad (4.5)$$

Hence, by K-upper continuity and the fact that $f(x', x') = 0$, for k large enough the right-hand side of (4.3) belongs to $V_Y - K$, while the left-hand side belongs to $f(\bar{x}, x') + V_Y$, contradicting (4.5).

Since $\bar{x} \in S$, from (iii), (4.4) implies that $x' \in S$ which contradicts $x' \in Q$. □

In order to see that Theorem 1 is indeed a generalization of Theorem 5.1 of Miglierina and Molho (2003), it is worthwhile to compare the assumptions of these two results. Assumption (ii) is always fulfilled in Theorem 5.1, since in case of vector optimization $\phi(x) = \min F(D) - F(x)$. Assumption (iii) follows immediately from the particular structure of the function $f(x, y) = F(y) - F(x)$, while assumption (iv) weakens the continuity assumption on the function F. The next proposition shows that assumption (v), that reduces to the boundedness of the sequence $\{F(x_n)\}$, is always fulfilled under the hypothesis of Theorem 5.1 of Miglierina and Molho (2003).

Proposition 5 *Let $D \subseteq \mathbb{R}^n$ be a closed convex set, Y be a real topological vector space ordered by a closed, convex cone K with nonempty interior, and let $F : D \to Y$ be a continuous function such that the set $S = Eff(F, D)$ is bounded. Then for any minimizing sequence $\{x_n\}$, the sequence $\{F(x_n)\}$ is bounded (in Y).*

Proof Notice that, from the assumptions, $\mathrm{cl}(S)$ is a compact set. Therefore, $F(\mathrm{cl}(S))$ is a compact subset of Y. Since (see Köthe 1969, p. 135) every compact subset of a topological vector space is bounded, then $F(S) \subset F(\mathrm{cl}(S))$ is bounded. From Proposition 2, and from the boundedness of $F(S)$, the thesis follows trivially. □

A sufficient condition for M-well-posedness of the vector equilibrium problem can be given more directly in terms of properties of the function f, replacing the assumption (ii) in Theorem 1 with a kind of pseudomonotonicity of f, that reduces to the usual pseudomonotonicity in the scalar case, and is always fullfilled for vector optimization problems. This condition [(ii) in Theorem 2 below], beside the fact that it is more explicit than condition (ii) of Theorem 1, does not imply that the set S should be a singleton as, for example, strict pseudomonotonicity used in Bednarczuk 2007 implies.

Notice that also condition (ii′) is automatically satisfied for any vector optimization problem. Indeed, in this case $\min_K f(x, D) = \min_K F(D) - F(x) = F(Eff(F, D)) - F(x) = F(S) - F(x) = f(x, S)$.

Theorem 2 *Let X be finite dimensional ($X = \mathbb{R}^n$), $D \subseteq X$ be a closed convex set, and $f : D \times D \to Y$ such that*

(i) *the solution set S of (VEP) is nonempty and compact;*
(ii) *for every $x, z \in D$ such that $f(x, z) \in K$ we have $f(z, x) \in -K$;*
(ii′) *for every $x \in D \setminus S$ it follows that $\min_K f(x, D) \subseteq f(x, S)$;*
(iii) *$f(x, z) \neq 0$ whenever $x \in S$ and $z \in D \setminus S$;*
(iv) *f is K-upper continuous on $D \times D$, and K-concave with respect to its first variable;*
(v) *For every maximizing sequence $\{x_n\} \subset D$ and every $z \in D$, the sequence $\{f(x_n, z)\}$ is bounded in Y.*

Then the problem (VEP) is M-well-posed.

Proof The proof goes in the same way as Theorem 1. If the sequence $\{x_n\}$ is unbounded, we argue as in the proof of Theorem 1, since (global) K-upper continuity implies K-upper continuity with respect to the first variable. If the sequence is bounded, then it has a convergent subsequence $\{x_{n_k}\}$ with limit $x^* \in D$. By relation (4.1) it is obvious that $x^* \notin S$. Since $\{x_n\}$ is maximizing, one can choose a sequence $u_n \in \phi(x_n) = \min_K f(x_n, D)$ with $u_n \to 0$. By (ii’) $\min_K f(x_n, D) \subseteq f(x_n, S)$, hence there exists a vector $z_n \in S$ such that $u_n = f(x_n, z_n)$. Since S is compact, one may suppose (passing to a subsequence, if necessary) that the sequence $\{z_{n_k}\}$ converges to $z^* \in S$. By K-upper continuity, we conclude that for every $V_Y \in \mathcal{V}_Y(0)$ there exists k_0 such that

$$u_{n_k} = f(x_{n_k}, z_{n_k}) \in f(x^*, z^*) + V_Y - K, \quad \forall k \geq k_0.$$

This implies that $f(x^*, z^*) \in K$ (if not, a similar argument as in the proof of Theorem 1 leads to a contradiction). Now by (ii) we obtain that $f(z^*, x^*) \in -K$ and this, together with $z^* \in S$ leads to $f(z^*, x^*) = 0$, contradicting (iii). □

The next result is a further generalization of Theorem 5.1 in Miglierina and Molho (2003). It is obvious that in the framework of Theorem 5.1 assumptions (iii) and (iv) below are satisfied, since the compactness of $S = Eff(F, D)$ implies $cl(S) = S$. Theorem 3 is somehow similar to Theorem 2, but some of their assumptions are quite different. Indeed, in Theorem 2 we require the compactness of the set S and a generalized monotonicity of the function f, which need not to be assumed in Theorem 3. On the other hand, in Theorem 3 we require continuity and a stronger demand on the behaviour of the function f [see assumption (iii)].

Theorem 3 *Let X be finite dimensional ($X = \mathbb{R}^n$), $D \subseteq X$ be a closed convex set, and $f : D \times D \to Y$ such that*

- (i) *the solution set S of (VEP) is nonempty and bounded;*
- (ii) *for every $x \in D \backslash cl(S)$ it follows that $\min_K f(x, D) \subseteq f(x, cl(S))$;*
- (iii) *$f(x, z) \neq 0$ and $f(z, x) \neq 0$ whenever $x \in cl(S)$ and $z \in D \backslash cl(S)$;*
- (iv) *f is continuous (on $D \times D$) and $x \mapsto f(x, z)$ is K-concave for every $z \in D$;*
- (v) *for every maximizing sequence $\{x_n\} \subset D$ and every $z \in D$, the sequence $\{f(x_n, z)\}$ is bounded in Y.*

Then the problem (VEP) is M-well-posed.

Proof The proof goes in the same way as Theorem 1. Indeed, denote by $\{x_n\}$ a maximizing sequence satisfying (4.1). If $\{x_n\}$ is unbounded, a slight change in the proof leads to a contradiction. If the sequence is bounded, it has a convergent subsequence $\{x_{n_k}\}$ with limit $x^* \in D$. Then we follow the proof of Theorem 2, by replacing S with $cl(S)$. Indeed, by relation (4.1) it is obvious that $x^* \notin cl(S)$. Since $\{x_n\}$ is maximizing, one can choose a sequence $u_n \in \phi(x_n) = \min_K f(x_n, D)$ with $u_n \to 0$. By (ii) $\min_K f(x_n, D) \subseteq f(x_n, cl(S))$, hence there exists a vector $z_n \in cl(S)$ such that $u_n = f(x_n, z_n)$. Since $cl(S)$ is compact, one may suppose (passing to a subsequence, if necessary) that the sequence $\{z_{n_k}\}$ converges to $z^* \in cl(S)$. Using now the continuity of f the latter together with $u_{n_k} \to 0$ leads to $f(x^*, z^*) = 0$. Since $z^* \in cl(S)$ and $x^* \notin cl(S)$ we obtain a contradiction to (iii). □

To conclude our discussion, we provide now a sufficient condition for B-well-posedness of (VEP).

Corollary 1 *Suppose that all the conditions of Theorem 3 are satisfied, and furthermore:*

(vi) *for every $V_Y \in \mathcal{V}_Y(0)$ there exists $\tilde{V}_Y \in \mathcal{V}_Y(0)$ such that*

$$f(D \backslash cl(S), cl(S)) \cap (K + \tilde{V}_Y) \subseteq V_Y. \tag{4.6}$$

Then, the vector equilibrium problem is B-well-posed.

Proof Since by Theorem 3 (VEP) is M-well-posed, we get the thesis by applying Proposition 4, if we verify (3.2). To do this, observe that by Theorem 3 (ii) we have

$$\phi(x) = \min_K f(x, D) \subseteq f(x, cl(S)), \quad \forall x \in D \backslash cl(S).$$

This leads to $\phi(D \setminus cl(S)) \subseteq f(D \setminus cl(S), cl(S))$, which, together with (4.6) proves the thesis. □

In the particular case of vector optimization, we get

$$f(D \setminus S, S) = F(S) - F(D \setminus S) \subseteq F(S) - F(D) = \min_K F(D) - F(D).$$

This shows that, assuming S closed, condition (4.6) is weaker than the condition of Proposition 3.5 in Miglierina et al. (2005).

More recently, the well-posedness of quasiconvex vector optimization problems was investigated via a scalarization approach (see Miglierina et al. 2005; Durea 2007).

The extension to equilibrium problems will be the subject of a further research.

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