Nonlinear Analysis 72 (2010) 460–468

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Well-posed equilibrium problems

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ARTICLE INFO

Article history:
Received 19 November 2007
Accepted 18 June 2009

MSC:
49K40
90C31

Keywords:
Equilibrium problem
Well-posedness
Approximate solution

ABSTRACT

In this paper we introduce some notions of well-posedness for scalar equilibrium problems in complete metric spaces or in Banach spaces. As equilibrium problem is a common extension of optimization, saddle point and variational inequality problems, our definitions originates from the well-posedness concepts already introduced for these problems.

We give sufficient conditions for two different kinds of well-posedness and show by means of counterexamples that these have no relationship in the general case. However, together with some additional assumptions, we show via Ekeland's principle for bifunctions a link between them.

Finally we discuss a parametric form of the equilibrium problem and introduce a well-posedness concept for it, which unifies the two different notions of well-posedness introduced in the first part.

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1. Introduction

Given a nonempty set D and a function $f : D \times D \to \mathbb{R}$, the problem of interest, called equilibrium problem (abbreviated EP) consists of finding an element $\bar{a} \in D$ such that $f(\bar{a}, b) \geq 0$, for every $b \in D$.

Equilibrium problems have been extensively studied in recent years (see for instance [1–6] and the references therein) especially due to their important particular cases such as optimization problems, minimax/saddle point problems, variational inequalities and Nash equilibrium problems. These problems are useful models of many practical problems arising in game theory, physics, economics, etc.

Well-posedness was considered in the past in connection with optimization problems. For a scalar optimization problem

$$\min h(a), \quad a \in D \tag{1.1}$$

where $h : D \to \mathbb{R}$, a sequence $\{a_n\}_n \subseteq D$ is said to be *minimizing* for the optimization problem when $h(a_n) \to \inf_D h$ as $n \to \infty$. Let us first recall a classical notion of well-posedness proposed by Tykhonov in [7] (see also [8]).

Definition 1. The optimization problem (1.1) is called *Tykhonov well-posed* if

(i) there exists a unique solution $\bar{a} \in D$ of (1.1);
(ii) every minimizing sequence converges to $\bar{a}$.

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doi:10.1016/j.na.2009.06.081

In case of maximization problems, the definition of Tykhonov well-posedness is modified using maximizing sequences in a straightforward way. Roughly speaking, the above concept means that points with values close to the value of the problem must be close to the (unique) solution. This property might be very useful when constructing algorithms aimed at solving the problem of interest.

Later, corresponding notions of well-posedness have been defined for other particular cases of EP, like saddle point problems and variational inequalities. Let us first recall the saddle point problem. Given two complete metric spaces X and Y, and $F : X \times Y \to \mathbb{R}$, the saddle point problem is to find a couple $(\bar{x}, \bar{y}) \in X \times Y$ such that

$$F(x, \bar{y}) \le F(\bar{x}, \bar{y}) \le F(\bar{x}, y), \quad \forall x \in X, y \in Y.$$

Solving a saddle point problem is equivalent to solving the following minimum problem:

$$\min \omega(x, y) = \min(\sup_x F(x, y) - \inf_y F(x, y)).$$

Since $\omega(x, y) \ge 0$ for every $(x, y) \in X \times Y$, this is equivalent to find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\omega(\bar{x}, \bar{y}) = 0$.

Reducing the saddle point problem to an optimization problem, the following definition follows naturally.

Definition 2 (*See [9]*). The saddle point problem is Tykhonov well-posed if

(i) there exists a unique saddle point $(\bar{x}, \bar{y}) \in X \times Y$;
(ii) every sequence (x_n, y_n) minimizing for ω converges to $(\bar{x}, \bar{y})$.

Recall that the diameter of a subset A of a metric space (D, d) is defined as

$$\text{diam}(A) = \sup\{d(a, b), a, b \in A\},$$

provided the set A is nonempty.

In a Banach space setting, under the assumptions of convexity and Gâteaux differentiability of the objective function h of (1.1), it was proved in [8] that Tykhonov well-posedness is equivalent to the condition

$$\text{diam}(\{a \in D : \langle \nabla h(a), b - a\rangle \ge -\epsilon\|a - b\|, \forall b \in D\}) \to 0, \quad \epsilon \downarrow 0.$$

This equivalence leads naturally to the definition of well-posedness for a general variational inequality: find $a \in D$ such that

$$\langle A(a), b - a\rangle \ge 0 \quad \forall b \in D, \tag{1.2}$$

where the map $A : D \to D^*$ is not necessarily a gradient map.

Definition 3 (*See [8]*). The variational inequality (1.2) is called well-posed if there exists at least one solution, and

$$\text{diam}(\{a \in D : \langle A(a), b - a\rangle \ge -\epsilon\|a - b\|, \forall b \in D\}) \to 0, \quad \epsilon \downarrow 0.$$

As mentioned before, optimization, saddle point and variational inequality problems constitute outstanding particular cases of EP. This fact leads naturally to the following questions: how can be assigned a proper definition of well-posedness to EP, which extends the above (existing) concepts? Which results concerning the new concept(s) can be achieved, and are they generalizations of known results? Our aim in this paper is to try to answer these questions. To do this, we define in Section 2 a concept of well-posedness for EP arising in a natural way from Definition 1 (Tychonov well-posedness for optimization problems). We call this concept T_{opt}-well-posedness and we show (Proposition 1) that both concepts of well-posedness, i.e. for optimization and saddle point problems (Definitions 1 and 2) can be obtained as its particular cases. Theorem 1 gives a characterization of T_{opt}-well-posedness in terms of *approximate solutions* of EP, while Proposition 3 provides sufficient conditions for T_{opt}-well-posedness.

Although the variational inequality problem is also a particular case of EP, it is interesting to quote that a natural extension of well-posedness from variational inequalities (cf. Definition 3) to EP defined in Section 3 and called T_{vi}-*well-posedness* leads to a different concept than T_{opt}-well-posedness. Theorem 2 provides alternative sufficient conditions, while Theorem 3 shows T_{vi}-well-posedness in a finite dimensional setting. The fact that the two concepts (T_{opt} and T_{vi}-well-posedness) for EP have no relationship in general is shown by means of Examples 1 and 2. However, under suitable additional hypotheses it turns out – involving an earlier result of the authors concerning Ekeland's principle for EP [10] – that T_{vi}-well-posedness implies T_{opt}-well-posedness (Corollary 2 in Section 4).

Finally, in Section 5 we deal with a parametric form of EP together with the so-called Hadamard well-posedness, which can be seen as a common extension of both T_{opt} and T_{vi}-well-posedness investigated in the previous sections. This allows us to obtain both T_{opt} and T_{vi}-well-posedness under the same hypotheses on EP (Corollary 3) by means of Hadamard well-posedness (Proposition 4).

2. Well-posedness coming from optimization

Unless otherwise stated, throughout this paper (D, d) is a complete metric space and $f : D \times D \to \mathbb{R}$ a given function such that $f(a, a) = 0$ for every $a \in D$. To start our analysis, we introduce a well-known minimax formulation of EP. This leads to consider the extended-valued gap function $g : D \to [-\infty, +\infty)$ defined by

$$g(a) = \inf_{b \in D} f(a, b).$$

Peculiar properties of the function g are the non-positivity on the set D, and the fact that $g(a^*) = 0$ if and only if a^* is a solution of EP. Therefore, the following result holds:

Lemma 1. *The equilibrium problem has solutions if and only if*

$$\max_{a \in D} g(a) = 0.$$

By the previous formulation of EP and Definition 1 we can provide a natural definition of Tykhonov well-posedness for equilibrium problems via the function g.

Definition 4. The equilibrium problem EP is T_{opt}-well-posed if

(i) there exists only one solution $\bar{a} \in D$ of EP;
(ii) for every sequence $\{a_n\} \subset D$ such that $g(a_n) \to 0$, it is $a_n \to \bar{a}$.

The sequence $\{a_n\}$ in (ii) is still called *maximizing* for g.

The definition given above entails, as particular cases, the notions of well-posedness for optimization and saddle point problems, as proved in the following proposition.

Proposition 1. (i) *If $f(a, b) = h(b) - h(a)$, then EP is T_{opt}-well-posed (in the sense of Definition 4) if and only if $\min_{b \in D} h(b)$ is Tykhonov well-posed (in the sense of Definition 1).*

(ii) *If $F : X \times Y \to \mathbb{R}$, $D = X \times Y$, $a = (x, y)$ and $b = (u, v)$, define $f : D \times D \to \mathbb{R}$ as $f(a, b) = F(x, v) - F(u, y)$. Then EP is T_{opt}-well-posed if and only if the saddle-point problem engendered by F is well-posed (in the sense of Definition 2).*

Let us give an example of EP that is T_{opt}-well-posed.

Example 1. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by $f(a, b) = -|b - a|a^2 e^{-a}$. The equilibrium problem associated is well-posed; indeed:

(i) $f(\bar{a}, b) \geq 0$ for every $b \in \mathbb{R}$ if and only if $\bar{a} = 0$;
(ii) $g(a) = \inf_{b \in \mathbb{R}} f(a, b) = -\sup_{b \in \mathbb{R}} |b - a|a^2 e^{-a} = \begin{cases} 0, & a = 0 \\ -\infty, & a \neq 0. \end{cases}$

Take a_n such that $g(a_n) \to 0$; from (ii), this means that $a_n = 0$ for n large enough.

T_{opt}-well-posedness can be investigated and characterized via the notion of some approximate solutions of EP. In order to show this, let $\epsilon > 0$ be given and let us introduce the set

$$\epsilon - \text{argmin}(EP) = \{a \in D : f(a, b) \geq -\epsilon, \forall b \in D\}.$$

The family of sets $\{\epsilon - \text{argmin}(EP)\}_\epsilon$ is ascending, i.e., if $\epsilon_1 < \epsilon_2$, then $\epsilon_1 - \text{argmin}(EP) \subseteq \epsilon_2 - \text{argmin}(EP)$. Moreover, the set of solutions of EP is the intersection of the sets $\{\epsilon - \text{argmin}(EP) : \epsilon > 0\}$.

The next result generalizes Theorem I.11 in [8] on one hand and provides an alternative characterization for T_{opt}-well-posedness, on the other hand.

Theorem 1. *If EP is T_{opt}-well-posed, then*

$$\text{diam}(\epsilon - \text{argmin}(EP)) \to 0, \quad \epsilon \downarrow 0. \tag{2.1}$$

Moreover, the converse is true if $a \mapsto f(a, b)$ is upper semicontinuous for every $b \in D$, and $\epsilon - \text{argmin}(EP)$ is nonempty for every $\epsilon > 0$.

Proof. By contradiction, assume that

$$\text{diam}(\epsilon - \text{argmin}(EP)) \nrightarrow 0.$$

Then there exist $\alpha > 0$ and $\epsilon_n \downarrow 0$ such that

$$\text{diam}(\epsilon_n - \text{argmin}(EP)) \geq 2\alpha, \quad \forall n \in \mathbb{N}.$$

Take $a_n, a'_n \in D$ such that $a_n, a'_n \in \epsilon_n - \text{argmin}(EP)$, and $d(a_n, a'_n) \geq \alpha$, for every $n \in \mathbb{N}$. From the inequalities

$$f(a_n, b) \geq -\epsilon_n, \qquad f(a'_n, b) \geq -\epsilon_n, \quad \forall n \in \mathbb{N},\ \forall b \in D,$$

we get that, for every $n \in \mathbb{N}$,

$$\inf_{b \in D} f(a_n, b) \geq -\epsilon_n, \qquad \inf_{b \in D} f(a'_n, b) \geq -\epsilon_n$$

i.e.,

$$0 \geq g(a_n) \geq -\epsilon_n, \qquad 0 \geq g(a'_n) \geq -\epsilon_n.$$

The inequalities above imply that $g(a_n) \to 0$ and $g(a'_n) \to 0$ if $n \to \infty$, and, by the assumptions, both sequences $\{a_n\}$ and $\{a'_n\}$ converge to the unique solution $\bar{a}$ of EP, a contradiction.

Conversely, let us first show that every maximizing sequence is convergent, or, equivalently, is a Cauchy sequence. Let $\{a_n\}$ be a maximizing sequence. By contradiction, assume that there exist $\{a_{n_k}\}$ and $\{a_{m_k}\}$, both subsequences of $\{a_n\}$, such that $d(a_{n_k}, a_{m_k}) > \alpha$, for some $\alpha > 0$ and for every $k \in \mathbb{N}$. From (2.1), we can take ϵ such that

$$\text{diam}(\epsilon - \text{argmin}(EP)) < \alpha.$$

Since both $\{a_{n_k}\}$ and $\{a_{m_k}\}$ are maximizing sequences, there exists $k_\epsilon \in \mathbb{N}$ such that for $k \geq k_\epsilon$ we have that

$$a_{n_k}, a_{m_k} \in \epsilon - \text{argmin}(EP),$$

therefore $d(a_{n_k}, a_{m_k}) < \alpha$, a contradiction. This shows that $\{a_n\}$ is convergent. Take any maximizing sequence $\{a_n\}$, denote by $\bar{a}$ its limit, and fix any $b \in D$. Then, by upper semicontinuity,

$$f(\bar{a}, b) \geq \limsup_{n \to \infty} f(a_n, b) \geq \limsup_{n \to \infty} g(a_n) = 0.$$

Since $b \in D$ was arbitrary, we conclude that $\bar{a}$ is a solution of EP. The uniqueness follows immediately from (2.1). □

For the sake of completeness we report the proof of the following proposition that is the scalar version of Proposition 1 in [10], and that entails the nonemptiness of $\epsilon - \text{argmin(EP)}$. We recall that the function $f : D \times D \to \mathbb{R}$ satisfies the "triangular inequality" (TI) if

$$f(a, b) \leq f(a, c) + f(c, b),$$

for all $a, b, c \in D$.

Proposition 2. *Suppose that the function f satisfies (TI). If there exists $\hat{b} \in D$ such that the function $a \mapsto f(a, \hat{b})$ is upper bounded, then $\epsilon - \text{argmin}(EP) \neq \emptyset$ for every $\epsilon > 0$.*

Proof. By the upper boundedness $a \mapsto f(a, \hat{b})$, for every $\epsilon > 0$ there exists $a_0 \in D$ such that

$$f(a_0, \hat{b}) - f(a, \hat{b}) + \epsilon > 0, \quad \forall a \in D.$$

This inequality, together with (TI), gives

$$-\epsilon < f(a_0, \hat{b}) - f(a, \hat{b}) \leq f(a_0, a), \quad \forall a \in D,$$

i.e., $a_0 \in \epsilon - \text{argmin(EP)}$. □

In what follows, we will call *forcing* a function $c : [0, +\infty) \to [0, +\infty)$ provided it is increasing, $c(0) = 0$, and $t > 0$ implies $c(t) > 0$ (see, for instance, [11]).

Definition 5. A function $f : D \times D \to \mathbb{R}$ is said to be *forcing pseudomonotone* if there exists a forcing function c such that

$$f(a, b) \geq 0 \Longrightarrow f(b, a) \leq -c(d(a, b)), \quad \forall a, b \in D.$$

Under suitable assumptions we can provide a sufficient condition for T_{opt}-well-posedness.

Proposition 3. *Assume that f is a forcing pseudomonotone function and that EP has at least one solution. Then EP is T_{opt}-well-posed.*

Proof. The uniqueness of the solution follows easily from the assumption of forcing pseudomonotonicity. Denote by $\bar{a}$ the solution of EP. Let $\{a_n\}$ be a maximizing sequence for g. We have

$$g(a_n) \leq f(a_n, \bar{a}) \leq -c(d(a_n, \bar{a})) \leq 0.$$

Since $g(a_n) \uparrow 0$, we conclude that $d(a_n, \bar{a}) \to 0$, thereby showing that $a_n \to \bar{a}$. □

3. Well-posedness coming from variational inequalities

An alternative definition of well-posedness for EP originates from the framework of variational inequalities. Inspired by Definition 3, we denote by $E(\epsilon)$ the set of the ϵ-equilibrium points already defined in [12], i.e.,

$$E(\epsilon) = \{a \in D : f(a, b) \geq -\epsilon d(a, b), \forall b \in D\},$$

and introduce the following

Definition 6. The equilibrium problem is T_{vi}-well-posed if

(i) there exists at least one solution $\bar{a} \in D$ of EP;
(ii) $\mathrm{diam}(E(\epsilon)) \to 0, \epsilon \downarrow 0$.

Notice that condition (ii) trivially implies the uniqueness of the solution.

In the following, we give sufficient conditions for T_{vi}-well-posedness.

Theorem 2. *Assume that f is forcing pseudomonotone and that the solution set of EP is nonempty. If at least one of the following conditions holds:*

(i) *the forcing function c is coercive, that is,* $\lim_{t\to+\infty} \frac{c(t)}{t} = +\infty$, *or*
(ii) *D is a Banach space and f is concave in its first variable,*

then EP is T_{vi}*-well-posed.*

Proof. Let $\bar{a}$ be any solution. Suppose that assumption (i) holds. We first show that $E(\epsilon)$ is bounded for any $\epsilon > 0$. Indeed, fix $\epsilon > 0$ arbitrarily and take any $a \in E(\epsilon)$; this implies that $f(a, \bar{a}) \geq -\epsilon d(a, \bar{a})$. Since $\bar{a}$ is a solution, we know that $f(\bar{a}, a) \geq 0$, and, from the forcing pseudomonotonicity, $f(a, \bar{a}) \leq -c(d(a, \bar{a}))$. Therefore, for every $a \in E(\epsilon)$,

$$-\epsilon d(a, \bar{a}) \leq -c(d(a, \bar{a})),$$

which implies for every $a \in E(\epsilon) \setminus \{\bar{a}\}$ that

$$\frac{c(d(a, \bar{a}))}{d(a, \bar{a})} \leq \epsilon. \tag{3.1}$$

This, together with the coercivity of c shows that $E(\epsilon)$ is bounded.

Now let us show that $\mathrm{diam}(E(\epsilon)) \to 0$. Supposing the contrary, there exist $\alpha > 0$, $\epsilon_n \downarrow 0$ and $a_n \in E(\epsilon_n)$ such that $d(a_n, \bar{a}) \geq \alpha$. Therefore, since c is a forcing function, by (3.1) we obtain

$$0 < c(\alpha) \leq c(d(a_n, \bar{a})) \leq \epsilon_n d(a_n, \bar{a}) \leq \epsilon_n \mathrm{diam}(E(\epsilon_n)) \leq \epsilon_n \mathrm{diam}(E(R)),$$

where $R > 0$ is an upper bound of the sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$. Since $E(R)$ is bounded, this relation leads to a contradiction if we let $n \to \infty$, thus proving the assertion.

Suppose now that (ii) holds, and set $d(a, b) = \|a - b\|$. By contradiction, if $\mathrm{diam}(E(\epsilon)) \nrightarrow 0$ as $\epsilon \downarrow 0$, then (as in the first part) there exist $\alpha > 0$, $\epsilon_n \downarrow 0$ and $a_n \in E(\epsilon_n)$ such that $\|a_n - \bar{a}\| \geq \alpha$. Let

$$M := \sup\{f(a, \bar{a}), a \in D, \|a - \bar{a}\| \geq \alpha\}. \tag{3.2}$$

From the forcing pseudomonotonicity of f, we easily get that $M < 0$. Indeed, since $\bar{a}$ is a solution, we obtain for every $a \in D$ with $\|a - \bar{a}\| \geq \alpha$ that

$$f(a, \bar{a}) \leq -c(\|a - \bar{a}\|) \leq -c(\alpha)$$

showing that $M \leq -c(\alpha) < 0$. Since $a_n \in E(\epsilon_n)$, $f(a_n, \bar{a}) \geq -\epsilon_n \|a_n - \bar{a}\|$, for every n. Taking into account the assumption of concavity of $f(\cdot, b)$ and (3.2), the following chain of inequalities hold:

$$\begin{aligned} -\epsilon_n \alpha &\leq \frac{\alpha}{\|a_n - \bar{a}\|} f(a_n, \bar{a}) \\ &= \frac{\alpha}{\|a_n - \bar{a}\|} f(a_n, \bar{a}) + \left(1 - \frac{\alpha}{\|a_n - \bar{a}\|}\right) f(\bar{a}, \bar{a}) \\ &\leq f\left(\frac{\alpha}{\|a_n - \bar{a}\|} a_n + \left(1 - \frac{\alpha}{\|a_n - \bar{a}\|}\right) \bar{a}, \bar{a}\right) \\ &\leq M. \end{aligned}$$

This is a contradiction, since $M < 0$ and $\epsilon_n \downarrow 0$. □

Theorem 3. *Assume that f is upper semicontinuous in its first variable and $E(\epsilon)$ is compact for some $\epsilon > 0$. If EP has a unique solution, then*

$$\operatorname{diam}(E(\epsilon)) \to 0, \quad \epsilon \downarrow 0.$$

Proof. Let $\bar{a}$ be the solution of EP. By contradiction, suppose that $\operatorname{diam}(E(\epsilon)) \nrightarrow 0$ if $\epsilon \downarrow 0$. Then, there exist $\alpha > 0$, $\epsilon_n \downarrow 0$ and $a_n \in E(\epsilon_n)$ such that $d(a_n, \bar{a}) \geq \alpha$. From the assumptions, there exists $n_0 \in \mathbb{N}$ such that $E(\epsilon_n)$ is compact for every $n \geq n_0$. Since $a_n \in E(\epsilon_n) \subseteq E(\epsilon_{n_0})$, taking, if necessary, a subsequence, we get that $a_n \to a^* \neq \bar{a}$, as $d(\bar{a}, a^*) \geq \alpha$. Let us show that a^* is a solution of EP, a contradiction. Indeed, fix $b \in D$; from $f(a_n, b) \geq -\epsilon_n d(a_n, b)$ and the boundedness of $\{a_n\}$, there exists $M_b > 0$ such that

$$f(a_n, b) \geq -\epsilon_n M_b, \quad \forall n \geq n_0.$$

In particular,

$$\limsup f(a_n, b) \geq 0.$$

From the upper semicontinuity of $f(\cdot, b)$ we get that a^* is a solution. □

In case D is a finite dimensional space, a similar proof as before provides the following statement:

Corollary 1. *Let $D = \mathbb{R}^n$. Assume that f is upper semicontinuous in its first variable and $E(\epsilon)$ bounded for some $\epsilon > 0$. If EP has a unique solution, then*

$$\operatorname{diam}(E(\epsilon)) \to 0, \quad \epsilon \downarrow 0.$$

4. Relationship between T_{opt} and T_{vi}

In the following we are interested in the comparison between the two definitions of well-posedness for EP introduced in previous sections. Notice that, in general, there are no relationships between T_{opt}-well-posed problems and T_{vi}-well-posed ones. Indeed, in Example 1 the problem is T_{opt}-well-posed but not T_{vi}-well-posed, since $E(\epsilon)$ is unbounded for every $\epsilon > 0$.

The following example provides an EP that is T_{vi}-well-posed but not T_{opt}-well-posed.

Example 2. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by

$$f(a, b) := \begin{cases} -\dfrac{|a|b^2}{(a^2+1)(b^2+1)}, & \text{if } a \neq b, \\ 0, \quad \text{if } a = b. \end{cases}$$

It is clear that the only solution of EP is $\bar{a} = 0$. Also,

$$\begin{aligned} g(a) &= \inf_{b\in\mathbb{R}} f(a, b) = \inf_{b\in\mathbb{R}\setminus\{a\}} -\frac{|a|b^2}{(a^2+1)(b^2+1)} \\ &= -\sup_{b\in\mathbb{R}\setminus\{a\}} \frac{|a|b^2}{(a^2+1)(b^2+1)} = -\frac{|a|}{a^2+1}. \end{aligned}$$

Take the sequence $a_n = n$. Since $\lim_{n\to+\infty} g(n) = 0$, we have that EP is not T_{opt}-well-posed. On the other hand, it is easy to see that

$$E(\epsilon) = \{a \in \mathbb{R} : f(a, b) \geq -\epsilon|b - a|, \forall b \in \mathbb{R}\} = \{0\}$$

for every $\epsilon > 0$. Indeed, fix $\epsilon > 0$. Then $0 \in E(\epsilon)$ is trivial, and suppose that for some $a \neq 0$ one has $a \in E(\epsilon)$. The latter implies that

$$-\frac{|a|b^2}{(a^2+1)(b^2+1)} \geq -\epsilon|b - a|, \quad \forall b \in \mathbb{R} \setminus \{a\}.$$

Letting $b \to a$ in this relation, we obtain

$$0 > -\frac{|a|^3}{(a^2+1)^2} \geq 0,$$

a contradiction. Thus, $\operatorname{diam}(E(\epsilon)) = 0$ and such EP is T_{vi}-well-posed.

To compare T_{opt} and T_{vi}-well-posedness in some circumstances, we should find a link between the sets $\epsilon -$ argmin(EP) and E(ϵ). The next result provides an inclusion that turns out to be useful for our aim. An analogous property holds for optimization problems (see [8], Ch.2, Sec.4). First of all, we need the following lemma, that was proved in Theorem 2 in [10] in the vector-valued case.

Lemma 2. *Let $a_0 \in \epsilon - \text{argmin}(EP)$. Assume that f satisfies condition (TI) and it is lower bounded and lower semicontinuous with respect to its second variable. Let $\lambda > 0$. Then there exists $\bar{a} \in D$ such that*

(i) $f(\bar{a}, a_0) \geq 0$;
(ii) $d(\bar{a}, a_0) \leq \lambda$;
(iii) $f(\bar{a}, a) + (\epsilon/\lambda)d(\bar{a}, a) > 0, \forall a \neq \bar{a}$.

In what follows, for every $A \subseteq D$, we denote by ${}^{\epsilon}A$ the set

$$ {}^{\epsilon}A = \{a' \in D : d(a', A) \leq \epsilon\}, $$

where

$$ d(a', A) = \inf\{d(a', a) : a \in A\}. $$

Theorem 4. *Assume that f satisfies condition (TI) and is lower bounded and lower semicontinuous with respect to its second variable. Then for every $\epsilon > 0$ one has*

$$ \epsilon^2 - \text{argmin}(EP) \subseteq {}^{\epsilon}E(\epsilon). $$

Proof. Let $a \in \epsilon^2 - \text{argmin}(EP)$. From Lemma 2, taking $\lambda = \epsilon$, there exists $\bar{a} \in D$ such that $f(\bar{a}, b) \geq -\epsilon d(\bar{a}, b)$, for every $b \in D$, i.e., $\bar{a} \in E(\epsilon)$. From (ii) in Lemma 2, $d(\bar{a}, a) \leq \epsilon$. In particular, $d(E(\epsilon), a) \leq \epsilon$. Since a is arbitrary in ϵ^2–argmin(EP), the proof is complete. □

Combining Theorem 1, Proposition 2 and Theorem 4, a first relation between T_{opt}-well-posedness and T_{vi}-well-posedness can be derived.

Corollary 2. *Assume that f satisfies the following assumptions:*

(i) *(TI) holds;*
(ii) *f is lower bounded and lower semicontinuous with respect to its second variable;*
(iii) *f is upper semicontinuous with respect to its first variable;*
(iv) *there exists $\hat{b}$ such that $f(\cdot, \hat{b})$ is upper bounded.*

Then

$$ T_{vi}\text{-well-posedness} \Longrightarrow T_{opt}\text{-well-posedness}. $$

5. Hadamard well-posedness

In this section we deal with a parametric form of an equilibrium problem and a related well-posedness, which can be seen as a common extension of both T_{opt} and T_{vi}-well-posedness investigated in the previous sections. Let $f : D \times D \times \mathcal{U} \to \mathbb{R}$, where $\mathcal{U} \subset E$, and D, E are metric spaces. For a given $\epsilon \in \mathcal{U}$ consider the following equilibrium problem EP_{ϵ}: find $\bar{a} \in D$ such that

$$ f(\bar{a}, b, \epsilon) \geq 0, \quad \forall b \in D. $$

Denote by $F(\epsilon)$ the solutions of EP_{ϵ}.

Assuming existence and uniqueness of the solution of EP_{ϵ_0}, with $\epsilon_0 \in \mathcal{U}$, we are interested in the investigation of continuous dependence of the solutions with respect to the data of the problem, i.e. the so-called Hadamard well-posedness.

Definition 7 (*See, for a Comparison, [13]*). EP_{ϵ} is said to be Hadamard well-posed at $\epsilon_0 \in \mathcal{U}$ if

(i) $F(\epsilon_0) = \{\bar{a}\}$,
(ii) for any $\epsilon_n \to \epsilon_0$, and any $a_n \in F(\epsilon_n)$, $\{a_n\}$ converges to $\bar{a}$.

If $E = [0, +\infty)$ and $f_0 : D \times D \to \mathbb{R}$ with $f_0(a, a) = 0$ for every $a \in D$, consider the following expressions for the function f:

(i) $f(a, b, \epsilon) = f_0(a, b) + \epsilon$,
(ii) $f(a, b, \epsilon) = f_0(a, b) + \epsilon d(a, b)$.

In the first case, $F(\epsilon) = \epsilon - \mathrm{argmin}(EP)$, while in the second one, $F(\epsilon) = E(\epsilon)$, where EP is defined by f_0. This observation suggests that within the above framework Hadamard well-posedness reduces to T_{opt}-well-posedness in case (i), and to T_{vi}-well-posedness in case (ii), if we take $\epsilon_0 = 0$. Indeed, assume that the representation (i) holds and suppose that EP_ϵ is Hadamard well-posed at 0. Let $\{a_n\}$ be a sequence in D such that

$$g_0(a_n) = \inf_{b \in D} f_0(a_n, b) \to 0.$$

Choose $\epsilon_n = -g_0(a_n) + 1/n > 0$ $(n \geq 1)$. Clearly $\epsilon_n \to 0$. Since by the trivial inequality $g_0(a_n) \geq -\epsilon_n$ for every $n \geq 1$ we obtain that $a_n \in F(\epsilon_n)$, thus $a_n \to \bar{a}$, the unique solution of EP. Hence EP is T_{opt}-well-posed.

Before proving the assertion concerning T_{vi}-well-posedness, let us observe that the Hadamard well-posedness (in the general case) implies that $\mathrm{diam}(F(\epsilon)) \to 0$ as $\epsilon \to \epsilon_0$. Indeed, assume, by contradiction, that $\mathrm{diam}(F(\epsilon_n)) \geq 2\alpha > 0$ for a suitable sequence $\epsilon_n \to \epsilon_0$. Consider two sequences $\{a_n\}$ and $\{b_n\}$ in $F(\epsilon_n)$ such that $d(a_n, b_n) \geq \alpha$. From the Hadamard well-posedness and the triangular inequality, we get

$$\alpha \leq d(a_n, b_n) \leq d(a_n, \bar{a}) + d(b_n, \bar{a}) \to 0 \quad n \to \infty,$$

a contradiction.

Now supposing that the representation (ii) holds, we obtain by the Hadamard well-posedness that $\mathrm{diam}E(\epsilon) \to 0$ as $\epsilon \to 0$, therefore EP is T_{vi}-well-posed.

Returning to problem EP_ϵ, it is interesting to remark that, in general, $\mathrm{diam}(F(\epsilon)) \to 0$ as $\epsilon \to \epsilon_0$ does not imply that this problem is Hadamard well-posed, as the following example shows:

Example 3. Let $D = E := \mathbb{R}$, $\mathcal{U} := [0, 1] \subset \mathbb{R}$ and $h : D \times \mathcal{U} \to \mathbb{R}$ be given by

$$h(a, \epsilon) = \begin{cases} (a-1)^2 & \text{if } \epsilon = 0, \\ 0 & \text{if } \epsilon \neq 0 \text{ and } 0 < a < \epsilon, \\ 1 & \text{if } \epsilon \neq 0 \text{ and } a \leq 0 \text{ or } a \geq \epsilon. \end{cases}$$

Define the function $f : D \times D \times \mathcal{U} \to \mathbb{R}$ by $f(a, b, \epsilon) := h(b, \epsilon) - h(a, \epsilon)$. Then (for $\epsilon_0 := 0$), we get $F(0) = \{1\}$ and $F(\epsilon) = (0, \epsilon)$ for $\epsilon > 0$. It is clear that the associated EP_ϵ is not Hadamard well-posed.

It is not surprising that the equilibrium problem in the example above fails to be Hadamard well-posed, since $h(\cdot, \epsilon)$ and $h(\cdot, 0)$ have no relationships at all; indeed, in this case, the set $F(\epsilon)$ $(\epsilon > 0)$ is far from $F(0) = \{1\}$. To give positive results of Hadamard well-posedness, a reasonable approach requires some continuity assumptions on the function f.

Proposition 4. *Assume that D is compact and $F(\epsilon_0) = \{\bar{a}\}$. If $f(\cdot, b, \cdot)$ is upper semicontinuous at (a, ϵ_0) for every $a, b \in D$, then EP_ϵ is Hadamard well-posed at ϵ_0.*

Proof. By contradiction, assume that there exists $\{\epsilon_n\}$ such that $\epsilon_n \to \epsilon_0$ and $a_n \in F(\epsilon_n)$ such that $d(a_n, \bar{a}) > \alpha > 0$ for some subsequence. Since D is compact, without loss of generality, we can suppose that $a_n \to a^*$. By upper semicontinuity, a^* is in $F(\epsilon_0)$. Since $a^* \neq \bar{a}$ we get a contradiction. □

By means of the representations given after Definition 7, the above result provides the following sufficient condition for both T_{opt} and T_{vi}-well-posedness (compare also with Theorems 1 and 3).

Corollary 3. *Let D be a compact subset of a complete metric space, and let $f_0 : D \times D \to \mathbb{R}$ with $f_0(a, a) = 0$ for every $a \in D$, such that $f_0(\cdot, b)$ is upper semicontinuous for every $b \in D$. If the EP associated to f_0 has a unique solution, then this problem is both T_{opt} and T_{vi}-well-posed.*

Proof. Let $E = [0, +\infty)$ and define the function $f : D \times D \times E \to \mathbb{R}$ according to (i) or (ii) described after Definition 7. In both cases $f(\cdot, b, \cdot)$ is upper semicontinuous at $(a, 0)$ for every $a, b \in D$. Then, by Proposition 4, EP_ϵ is Hadamard well-posed at $\epsilon_0 = 0$ and taking into account the observations after Definition 7 it follows that the EP associated to f_0 is both T_{opt} and T_{vi}-well-posed. □

Acknowledgement

The research of the second author was supported by the grant PN.II, ID_523/2007.

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